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On the Classical $W_4^{(2)}$ Algebra

L. Chao

CCAST(World Laboratory), P.O.Box 8730, Beijing 100080, China.

Institute of Modern Physics, Northwest University,
Xian 710069, China

and

Q. P. Liu

CCAST(World Laboratory), P.O.Box 8730, Beijing 100080, China.

Department of Mathematics, Beijing Graduate School,
China University of Mining and Technology,
Beijing 100083, China¹

ABSTRACT We consider the classical $W_4^{(2)}$ algebra from the integrable system viewpoint. The integrable evolution equations associated with the $W_4^{(2)}$ algebra are constructed and the Miura maps, consequently modifications, are presented. Modifying the Miura maps, we give a free field realization the classical $W_4^{(2)}$ algebra. We also construct the Toda type integrable systems for it.

¹Mailing Address

1 INTRODUCTION

Integrable systems of nonlinear differential equations are studied extensively during last three decades[2]. These are the equations which possess remarkable analytical, geometric and algebraic properties. Evermore remarkable, this theory brings a number of different research fields together and finds applications in several branches. The interaction between integrable system theory and W algebra theory is just one of the fascinating points, which attracts much attention recently [1][3][14].

W_n algebra, which is higher spin generalization of Virasoro algebra, is introduced by Zamolodchikov [3] recently. It plays an important role in the theory of the two dimensional quantum gravity and matrix models. W_n algebra may be constructed via Hamiltonian reduction approach from the WZW model[1]. Novel W algebras exist which involves fields with fractional spins. These are referred to $W_n^{(l)}$ algebras. The first such example is the $W_3^{(2)}$ algebra of the Polyakov-Bershadsky [4], which consists of four fields: energy - momentum tensor, two bosonic fields of spin $\frac{3}{2}$ and a spin 1 $U(1)$ current.

Gervais[14] is the first to notice the interrelation between the KdV equation and the Virasoro algebra. Precisely, the second Poisson bracket of the KdV equation is equivalent to the classical Virasoro algebra. This result is generalized later on and the equivalence of the classical W_n algebra and the second Poisson bracket of the Gelfand-Dickey hierarchy is discovered[12][13]. This remarkable connection provides new insight into both theories. For example, constructing new type of integrable systems may lead to new type of W algebras, and vice versa. Another important point is that Miura map, which plays a central role in the Soliton theory, often provides free field realization for the corresponding W algebra. Noticing the connection of the W algebras and integrable evolution equations, it is not surprising that a correspondence between the W algebras and Toda type system exists. We refer the papers[1] for more details.

The present paper is on the $W_4^{(2)}$ algebra. Generally speaking, $W_n^{(l)}$ algebras are given by different $sl(2)$ embedding into $sl(n)$. The form of the $W_4^{(2)}$ algebra is inferred in [5], but its explicit form is presented by Bakas and Depireux[7] by means of Hamiltonian reduction method. We notice that Mathieu and Depireux[8] discussed the $W_n^{(l)}$ algebra from the integrable system viewpoint, but they failed to construct the integrable systems associated with the classical $W_4^{(2)}$ algebra. We construct such systems in this paper. Both nonlinear evolution equations and Toda type of systems will be given explicitly for the $W_4^{(2)}$ algebra. Also, by working out Miura maps, we obtain a free field realization for the classical $W_4^{(2)}$ algebra. A by-product is that the free field realization for the W algebra associated with matrix Schrodinger operator.

The paper is arranged as follows. We recall the explicit form of the classical $W_4^{(2)}$ algebra in the next section. In section three, we construct the hierarchy of nonlinear evolution equations for this algebra. Section four is intended to construct its free field realization. The integrable systems of Toda type are presented in section 5. Final section contains some comments.

2 THE CLASSICAL $W_4^{(2)}$ ALGEBRA

We recall the classical $W_4^{(2)}$ algebra in this section. It is presented by Bakas and Depireux[7] by means of Hamiltonian reduction approach. It reads

$$\begin{aligned}
\{T(x), T(y)\} &= (\partial^3 + T\partial + \partial T)\delta(x - y), \\
\{T(x), v(y)\} &= (\tfrac{1}{2}\partial^3 - \tfrac{3}{2}H\partial^2 + v\partial + \partial v + w\partial p - p\partial w - H_{xx} - 2H_x\partial)\delta(x - y), \\
\{T(x), q(y)\} &= (\tfrac{1}{2}\partial^2 w + q\partial + \partial q)\delta(x - y), \quad \{T(x), r(y)\} = (\tfrac{1}{2}\partial^2 p + r\partial + \partial r)\delta(x - y), \\
\{T(x), w(y)\} &= w\partial\delta(x - y), \quad \{T(x), p(y)\} = p\partial\delta(x - y), \quad \{T(x), H(y)\} = H\partial\delta(x - y), \\
\{v(x), v(y)\} &= (\tfrac{3}{4}\partial^3 - \tfrac{3}{4}H\partial H + v\partial + \partial v - \tfrac{3}{4}H_{xx} - \tfrac{3}{2}H_x\partial)\delta(x - y), \\
\{v(x), q(y)\} &= (\tfrac{3}{4}\partial^2 w + \tfrac{3}{4}H\partial w + \partial q + Hq + wv)\delta(x - y), \\
\{v(x), r(y)\} &= (-p\partial^2 - \tfrac{1}{4}\partial^2 p + p\partial H - pv - \tfrac{1}{4}H\partial p + r\partial - rH)\delta(x - y), \\
\{v(x), w(y)\} &= -q\delta(x - y), \quad \{v(x), p(y)\} = (-p\partial + r)\delta(x - y), \\
\{v(x), H(y)\} &= \tfrac{1}{2}(-\partial - H)\partial\delta(x - y), \quad \{q(x), q(y)\} = -\tfrac{3}{4}w\partial w\delta(x - y), \\
\{q(x), r(y)\} &= (\partial^3 - \partial^2 H - H\partial^2 + \partial v + u\partial - H(u + v) + H\partial H + \tfrac{1}{4}w\partial p)\delta(x - y), \\
\{q(x), w(y)\} &= 0, \quad \{q(x), p(y)\} = ((\partial - H)\partial - v + u)\delta(x - y), \\
\{q(x), H(y)\} &= (\tfrac{1}{2}w\partial - q)\delta(x - y), \quad \{r(x), r(y)\} = -\tfrac{3}{4}p\partial p\delta(x - y), \\
\{r(x), w(y)\} &= ((\partial - H)\partial + v - u)\delta(x - y), \quad \{r(x), p(y)\} = 0, \\
\{r(x), H(y)\} &= \tfrac{1}{2}(-p\partial + r)\delta(x - y), \quad \{w(x), w(y)\} = 0, \\
\{w(x), p(y)\} &= -2(\partial - H)\delta(x - y), \quad \{w(x), H(y)\} = -w\delta(x - y), \\
\{p(x), p(y)\} &= 0, \quad \{p(x), H(y)\} = p\delta(x - y), \quad \{H(x), H(y)\} = -\partial\delta(x - y).
\end{aligned} \tag{2.1}$$

We easily see that the algebra (2.1) is the one presented in[7] up to an invertible transformation.

3 THE INTEGRABLE HIERARCHY OF EVOLUTION EQUATIONS

In order to derive an integrable hierarchy of nonlinear evolution equations associated to the $W_4^{(2)}$ algebra, we specify the associated spectral problem first. Our spectral problem is

$$\Phi_x = \begin{bmatrix} 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \\ u & q & H + h & 0 \\ r & v & p & -H + h \end{bmatrix} \Phi \equiv U\Phi, \tag{3.1}$$

we adjoin (3.1) as usual with the time evolution of the wave function Φ : $\Phi_t = V\Phi$, then calculating the zero curvature equation: $U_t - V_x + [U, V] = 0$. By suitable adjustment, we find that the hierarchy related with (3.1) is represented as

$$f_{t_n} = \{f, \mathcal{H}_{n+1}\}_0 = \{f, \mathcal{H}_n\}_1, \quad f = u, v, q, r, p, w, H, h. \tag{3.2}$$

where two Poisson brackets are defined by

$$\begin{aligned}
\{u(x), u(y)\}_0 &= -2\partial\delta(x - y), & \{u(x), q(y)\}_0 &= w\delta(x - y), \\
\{u(x), r(y)\}_0 &= -p\delta(x - y), & \{v(x), v(y)\}_0 &= -2\partial\delta(x - y), \\
\{v(x), q(y)\}_0 &= -w\delta(x - y), & \{v(x), r(y)\}_0 &= p\delta(x - y), \\
\{q(x), r(y)\}_0 &= -2(\partial + H)\delta(x - y), & \text{all other brackets vanish.}
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
\{u(x), u(y)\}_1 &= (\partial^3 - (H + h)\partial(H + h) + u\partial + \partial u + (H + h)_{xx} + 2(H + h)_x\partial)\delta(x - y), \\
\{u(x), v(y)\}_1 &= (-w\partial p - wr + qp)\delta(x - y), \\
\{u(x), q(y)\}_1 &= (-w\partial^2 - w\partial(H + h) - wu + q\partial + q(H + h))\delta(x - y), \\
\{u(x), r(y)\}_1 &= ((\partial - H - h)(\partial p + r) + up)\delta(x - y), \\
\{u(x), w(y)\}_1 &= (-w\partial + q)\delta(x - y), \quad \{u(x), p(y)\}_1 = -r\delta(x - y), \\
\{u(x), H(y)\}_1 &= \frac{1}{2}(\partial - H - h)\partial\delta(x - y), \quad \{u(x), h(y)\}_1 = \frac{1}{2}(\partial - H - h)\partial\delta(x - y), \\
\{v(x), v(y)\}_1 &= (\partial^3 - (H - h)\partial(H - h) + v\partial + \partial v - (H - h)_{xx} - 2(H - h)_x\partial)\delta(x - y), \\
\{v(x), q(y)\}_1 &= ((\partial + H - h)(\partial w + q) + wv)\delta(x - y), \\
\{v(x), r(y)\}_1 &= (-p\partial^2 + p\partial(H - h) - pv + r(\partial - H + h))\delta(x - y), \\
\{v(x), w(y)\}_1 &= -q\delta(x - y), \quad \{v(x), p(y)\}_1 = (-p\partial + r)\delta(x - y), \\
\{v(x), H(y)\}_1 &= -\frac{1}{2}(\partial + H - h)\partial\delta(x - y), \quad \{v(x), h(y)\}_1 = \frac{1}{2}(\partial + H - h)\partial\delta(x - y), \\
\{q(x), q(y)\}_1 &= -w\partial w\delta(x - y), \\
\{q(x), r(y)\}_1 &= (\partial^3 - \partial^2 H - H\partial^2 + h_{xx} + 2h_x\partial + \partial v + u\partial - H(u + v) - h(v - u) + \\
&\quad + (H + h)\partial(H - h))\delta(x - y), \quad \{q(x), w(y)\}_1 = 0, \\
\{q(x), p(y)\}_1 &= ((\partial - H - h)\partial - v + u)\delta(x - y), \quad \{q(x), H(y)\}_1 = (\frac{1}{2}w\partial - q)\delta(x - y), \\
\{q(x), h(y)\}_1 &= -\frac{1}{2}w\partial\delta(x - y), \quad \{r(x), r(y)\}_1 = -p\partial p\delta(x - y), \\
\{r(x), w(y)\}_1 &= ((\partial + H - h)\partial + v - u)\delta(x - y), \quad \{r(x), p(y)\}_1 = 0, \\
\{r(x), H(y)\}_1 &= (-\frac{1}{2}p\partial + r)\delta(x - y), \quad \{r(x), h(y)\}_1 = -\frac{1}{2}p\partial\delta(x - y), \\
\{w(x), w(y)\}_1 &= 0, \quad \{w(x), p(y)\}_1 = 2(-\partial + H)\delta(x - y), \\
\{w(x), H(y)\}_1 &= -w\delta(x - y), \quad \{w(x), h(y)\}_1 = 0, \quad \{p(x), p(y)\}_1 = 0, \\
\{p(x), H(y)\}_1 &= p\delta(x - y), \quad \{p(x), h(y)\}_1 = 0, \\
\{H(x), H(y)\}_1 &= -\partial\delta(x - y), \quad \{H(x), h(y)\}_1 = 0, \quad \{h(x), h(y)\}_1 = -\partial\delta(x - y),
\end{aligned} \tag{3.4}$$

and Hamiltonians may be calculated from

$$\mathcal{H}_n = \frac{2}{n} \int \text{tr res}(L^{\frac{n}{2}})dx, \quad \forall n \geq 1 \tag{3.5}$$

where tr . and res mean taking matrix trace and the coefficient of the term ∂^{-1} respectively,

$$\mathbf{L} = \partial^2 - \begin{bmatrix} H + h & w \\ p & -H + h \end{bmatrix} \partial + \begin{bmatrix} u & q \\ r & v \end{bmatrix}. \tag{3.6}$$

Remark:

The hierarchy may be rewritten by means of Hamiltonian operator terminology: $\mathbf{u}_{t_n} = \mathbf{B}_0\delta(\mathcal{H}_n) = \mathbf{B}_1\delta(\mathcal{H}_{n+1})$ with \mathbf{B}_0 and \mathbf{B}_1 are 8×8 matrix differential operators, which may read off from the formulae(3.3-4).

Noticing another form of the spectral problem(3.1), which is nothing but the matrix Schrodinger problem, we may use the standard theory[15][6] to derive the corresponding Poisson structures. It is straightforward to check that the brackets(3.3) and (3.4) are the exactly the Gelfand-Dikii's first and second brackets.

The classical $W_4^{(2)}$ algebra comes into play with the following observation: If we do the reduction $h = 0$ for the Poisson algebra(3.4), we obtain

$$\begin{aligned}
\{u(x), u(y)\} &= (\frac{3}{4}\partial^3 - \frac{3}{4}H\partial H + u\partial + \partial u + \frac{3}{4}H_{xx} + \frac{3}{2}H_x\partial)\delta(x - y), \\
\{u(x), v(y)\} &= (-w\partial p - wr + qp - \frac{1}{4}(\partial^3 - \partial^2 H - H\partial^2 + H\partial H))\delta(x - y), \\
\{u(x), q(y)\} &= (-w\partial^2 - w\partial H - wu + q\partial + qH - \frac{1}{4}(\partial - H)\partial w))\delta(x - y), \\
\{u(x), r(y)\} &= ((\partial - H)(\partial p + r) + up - \frac{1}{4}(\partial - H)\partial p))\delta(x - y), \\
\{u(x), w(y)\} &= (-w\partial + q)\delta(x - y), \quad \{u(x), p(y)\} = -r\delta(x - y), \\
\{u(x), H(y)\} &= \frac{1}{2}(\partial - H)\partial\delta(x - y), \\
\{v(x), v(y)\} &= (\frac{3}{4}\partial^3 - \frac{3}{4}H\partial H + v\partial + \partial v - \frac{3}{4}H_{xx} - \frac{3}{2}H_x\partial)\delta(x - y), \\
\{v(x), q(y)\} &= (\frac{3}{4}\partial^2 w + \frac{3}{4}H\partial w + \partial q + Hq + wv)\delta(x - y), \\
\{v(x), r(y)\} &= -(p\partial^2 + \frac{1}{4}\partial^2 p - p\partial H + pv + \frac{1}{4}H\partial p - r\partial + rH)\delta(x - y), \\
\{v(x), w(y)\} &= -q\delta(x - y), \quad \{v(x), p(y)\} = (-p\partial + r)\delta(x - y), \\
\{v(x), H(y)\} &= -\frac{1}{2}(\partial + H)\partial\delta(x - y), \quad \{q(x), q(y)\} = -\frac{3}{4}w\partial w\delta(x - y), \\
\{q(x), r(y)\} &= (\partial^3 - \partial^2 H - H\partial^2 + \partial v + u\partial - H(u + v) + H\partial H + \frac{1}{4}w\partial p)\delta(x - y), \\
\{q(x), w(y)\} &= 0, \quad \{q(x), p(y)\} = (\partial - H)\partial - v + u)\delta(x - y), \\
\{q(x), H(y)\} &= (\frac{1}{2}w\partial - q)\delta(x - y), \quad \{r(x), r(y)\} = -\frac{3}{4}p\partial p\delta(x - y), \\
\{r(x), w(y)\} &= ((\partial - H)\partial + v - u)\delta(x - y), \quad \{r(x), p(y)\} = 0, \\
\{r(x), H(y)\} &= -\frac{1}{2}(p\partial - r)\delta(x - y), \quad \{w(x), w(y)\} = 0, \\
\{w(x), p(y)\} &= 2(-\partial + H)\delta(x - y), \quad \{w(x), H(y)\} = -w\delta(x - y). \\
\{p(x), p(y)\} &= 0, \quad \{p(x), H(y)\} = p\delta(x - y), \quad \{H(x), H(y)\} = -\partial\delta(x - y),
\end{aligned} \tag{3.7}$$

This algebra is nothing but the classical $W_4^{(2)}$ algebra(2.1) with the fields redefinition

$$T = u + v - H^2 - pw, \quad v = v, \quad p = p, \quad w = w, \quad H = H, \quad q = q, \quad r = r. \tag{3.8}$$

Thus, we rediscover the classical $W_4^{(2)}$ algebra from the viewpoint of integrable systems. Because of this equivalence, we call the Poisson algebra(3.7) $W_4^{(2)}$ also. The explicit form of integrable hierarchy associated with it can be read off from the hierarchy(3.2). Here, we just give the first

non trivial flow

$$\begin{aligned}
u_t &= \frac{1}{2}(-HH_x + H_{xx} + 2u_x - wp_x - wr + qp), \\
v_t &= \frac{1}{2}(-HH_x + H_{xx} + 2v_x - w_x p + wr + qp), \\
q_t &= \frac{1}{2}(-Hw_x + w_{xx} + 2q_x + wH_x - 2Hq + wu - wv), \\
r_t &= \frac{1}{2}(Hp_x + p_{xx} + 2r_x - pH_x - pu + pv + 2rH), \\
w_t &= p_t = H_t = 0.
\end{aligned} \tag{3.9}$$

Remark:

We note that in the system(3.9) the time evolution of the fields (w,p,H) is trivial. This means that the dynamical system may be reduced to the submanifold of (u,v,q,r). In fact, this is a general phenomenon: the whole hierarchy(3.2) is reducible to the submanifold (u,v,q,r).

4 THE FREE FIELD REALIZATION OF THE $W_4^{(2)}$ ALGEBRA

For a given W algebra, it is important yet interesting to construct free field realization. Next we construct such realization for our $W_4^{(2)}$ algebra(3.7). To this end, we start with the derivation of Miura maps for the related hierarchy.

Let us do the following factorization

$$\mathbf{L} = (\partial - M)(\partial - N) \tag{4.1}$$

where \mathbf{L} is given by (3.6), $M = \begin{bmatrix} g_1 & k \\ l & g_2 \end{bmatrix}$, $N = \begin{bmatrix} m_1 & n \\ s & m_2 \end{bmatrix}$. Then, the transformation between field variables, which is a Miura map, reads

$$\begin{aligned}
u &= g_1 m_1 + ks - m_{1x}, & v &= \ln + g_2 m_2 - m_{2x}, \\
q &= g_1 n + km_2 - n_x, & r &= lm_1 + g_2 s - s_x, & w &= k + n, \\
p &= l + s, & H &= \frac{1}{2}(g_1 + m_1 - g_2 - m_2), & h &= \frac{1}{2}(g_1 + g_2 + m_1 + m_2),
\end{aligned} \tag{4.2}$$

and the spectral problem for the modified hierarchy is

$$\Psi_x = \begin{bmatrix} m_1 & n & \lambda & 0 \\ s & m_2 & 0 & \lambda \\ \lambda & 0 & g_1 & k \\ 0 & \lambda & l & g_2 \end{bmatrix} \Psi. \tag{4.3}$$

As for the modified Poisson bracket, one may either calculate the bracket directly following[15][6] or use (4.3) to calculate zero curvature equation. The resulted bracket is defined by the Hamiltonian operator

$$\hat{\mathbf{B}}_0 = \begin{bmatrix} -\partial & 0 & n & -s & 0 & 0 & 0 & 0 \\ 0 & -\partial & -n & s & 0 & 0 & 0 & 0 \\ -n & n & 0 & -\partial + m_1 - m_2 & 0 & 0 & 0 & 0 \\ s & -s & -\partial - m_1 + m_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\partial & 0 & k & -l \\ 0 & 0 & 0 & 0 & 0 & -\partial & -k & l \\ 0 & 0 & 0 & 0 & -k & k & 0 & -\partial + g_1 - g_2 \\ 0 & 0 & 0 & 0 & l & -l & -\partial - g_1 + g_2 & 0 \end{bmatrix}, \tag{4.4}$$

it can be directly verified that the Miura map(4.2) is a Hamiltonian or Poisson map. That is, it maps the modified Poisson bracket, defined by(4.4), to the Poisson bracket(3.4). Up to now, all these are known(see[15][6]). However, we note that unlike the scalar case, the Miura map(4.2) does not supply us a free field realization for the Poisson algebra(3.4) although it does simplify this algebra. To obtain such a realization, we need to introduce further coordinates transformations. Since the block structure of the $\hat{\mathbf{B}}_0$, we only need to work on the subspace(m_1, m_2, n, s) with

$$\mathbf{B}_{11} = \begin{bmatrix} -\partial & 0 & n & -s \\ 0 & -\partial & -n & s \\ -n & n & 0 & -\partial + m_1 - m_2 \\ s & -s & -\partial - m_1 + m_2 & 0 \end{bmatrix}, \quad (4.5)$$

we observe that the following transformation

$$\bar{m}_1 = m_1 + m_2, \quad \bar{m}_2 = m_1 - m_2, \quad \bar{n} = n, \quad \bar{s} = s, \quad (4.6)$$

maps (4.5) to

$$\hat{\mathbf{B}}_{11} = \begin{bmatrix} -2\partial & 0 & 0 & 0 \\ 0 & -2\partial & 2\bar{n} & -2\bar{s} \\ 0 & -2\bar{n} & 0 & -\partial + \bar{m}_2 \\ 0 & 2\bar{s} & -\partial - \bar{m}_2 & 0 \end{bmatrix}. \quad (4.7)$$

At this stage, we may use Wakimoto construction[17] to simplify the structure(4.7) further. Thus, we have

$$\bar{m}_1 = \xi, \quad \bar{m}_2 = \sqrt{2}\alpha + 2\beta\gamma, \quad \bar{n} = -\beta\gamma^2 + \gamma_x - \sqrt{2}\gamma\alpha, \quad \bar{s} = \beta, \quad (4.8)$$

and the operator in this coordinate $(\xi, \alpha, \gamma, \beta)$ is

$$\mathbf{D}_{11} = \begin{bmatrix} -2\partial & 0 & 0 & 0 \\ 0 & -\partial & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \quad (4.9)$$

Since the block structure of the operator(4.4), we only need to do the exact same transformations(4.6) and (4.9) for the other block. That is,

$$\bar{g}_1 = g_1 + g_2, \quad \bar{g}_2 = g_1 - g_2, \quad \bar{k} = k, \quad \bar{l} = l \quad (4.10)$$

$$\bar{g}_1 = \zeta, \quad \bar{g}_2 = \sqrt{2}\mu + 2\eta\nu, \quad \bar{k} = -\nu\eta^2 + \eta_x - \sqrt{2}\mu\eta, \quad \bar{s} = \nu. \quad (4.11)$$

Then, the final bracket is defined by the operator in the coordinate $(\xi, \alpha, \gamma, \beta, \zeta, \mu, \eta, \nu)$

$$\mathbf{D} = \begin{bmatrix} -2\partial & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\partial & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\partial & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\partial & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (4.12)$$

Thus, we reach the free field realization for the Poisson algebra(3.4).

Now we turn to the classical $W_4^{(2)}$ algebra. Let us first do the recoordinating

$$(m_1, m_2, n, s, g_1, g_2, k, l) \rightarrow (m_1, m_2, n, s, g_1, k, l, E) \quad (4.13)$$

where $E = m_1 + m_2 + g_1 + g_2$. Under this new coordinate, $\hat{\mathbf{B}}_0$ becomes

$$\bar{\mathbf{B}}_0 = \begin{bmatrix} -\partial & 0 & n & -s & 0 & 0 & 0 & -\partial \\ 0 & -\partial & -n & s & 0 & 0 & 0 & -\partial \\ -n & n & 0 & -\partial + m_1 - m_2 & 0 & 0 & 0 & 0 \\ s & -s & -\partial - m_1 + m_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\partial & k & -l & -\partial \\ 0 & 0 & 0 & 0 & -k & 0 & -\partial + g_1 - g_2 & 0 \\ 0 & 0 & 0 & 0 & l & -\partial - g_1 + g_2 & 0 & 0 \\ -\partial & -\partial & 0 & 0 & -\partial & 0 & 0 & -4\partial \end{bmatrix} \quad (4.14)$$

where $g_2 = E - m_1 - m_2 - g_1$.

Now we do the Dirac reduction $E = 0$ for the structure $\bar{\mathbf{B}}_0$, the resulted structure is

$$\hat{\mathbf{B}}_0 =$$

$$\begin{bmatrix} -\frac{3}{4}\partial & \frac{1}{4}\partial & n & -s & \frac{1}{4}\partial & 0 & 0 \\ \frac{1}{4}\partial & -\frac{3}{4}\partial & -n & s & \frac{1}{4}\partial & 0 & 0 \\ -n & n & 0 & -\partial + m_1 - m_2 & 0 & 0 & 0 \\ s & -s & -\partial - m_1 + m_2 & 0 & 0 & 0 & 0 \\ \frac{1}{4}\partial & \frac{1}{4}\partial & 0 & 0 & -\frac{3}{4}\partial & k & -l \\ 0 & 0 & 0 & 0 & -k & 0 & -\partial + 2g_1 + m_1 + m_2 \\ 0 & 0 & 0 & 0 & l & -\partial - 2g_1 - m_1 - m_2 & 0 \end{bmatrix} \quad (4.15)$$

We claim that the Poisson structure induced by $\hat{\mathbf{B}}_0$ is related to the algebra $W_4^{(2)}$ (3.7) by the following transformation

$$\begin{aligned} u &= g_1 m_1 + k s - m_{1x}, & v &= l n - m_2 (m_1 + m_2 + g_1) - m_{2x}, \\ q &= g_1 n + k m_2 - n_x, & r &= l m_1 - s (g_1 + m_1 + m_2) - s_x \\ w &= k + n, & p &= l + s, & H &= g_1 + m_1 \end{aligned} \quad (4.16)$$

This statement may be verified by tedious but straightforward calculation.

As above, this realization does not qualify as a free field realization and we need simplify (4.15) further to reach such position. Our observation is that the following coordinations transformation $(m_1, m_2, n, s, g_1, k, l) \rightarrow (\hat{m}_1, \hat{m}_2, \hat{n}, \hat{s}, \hat{g}_1, \hat{k}, \hat{l})$

$$\begin{aligned} \hat{m}_1 &= m_1 + m_2, & \hat{m}_2 &= m_1 - m_2, & \hat{n} &= n, & \hat{s} &= s, \\ \hat{g}_1 &= 2g_1 + m_1 + m_2, & \hat{k} &= k, & \hat{l} &= l, \end{aligned} \quad (4.17)$$

brings $\bar{\mathbf{B}}_0$ into a block matrix operator:

$$\tilde{B}_0 = \begin{bmatrix} -\partial & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\partial & 2\hat{n} & -2\hat{s} & 0 & 0 & 0 \\ 0 & -2\hat{n} & 0 & -\partial + \hat{m}_2 & 0 & 0 & 0 \\ 0 & 2\hat{s} & -\partial - \hat{m}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\partial & 2\hat{k} & -2\hat{l} \\ 0 & 0 & 0 & 0 & -2\hat{k} & 0 & -\partial + \hat{g}_1 \\ 0 & 0 & 0 & 0 & 2\hat{l} & -\partial - \hat{g}_1 & 0 \end{bmatrix} \quad (4.18)$$

Interesting enough, once again, we may use the Wakimoto construction directly for the structure(4.18). It reads as

$$\begin{aligned} \hat{m}_1 &= \theta, & \hat{m}_2 &= \sqrt{2}\theta_1 + 2\theta_2\theta_3, & \hat{n} &= -\theta_2^2\theta_3 + \theta_{2x} - \sqrt{2}\theta_1\theta_2, & \hat{s} &= \theta_3, \\ \hat{g}_1 &= \sqrt{2}\vartheta_1 + 2\vartheta_2\vartheta_3, & \hat{k} &= -\vartheta_2^2\vartheta_3 + \vartheta_{2x} - \sqrt{2}\vartheta_1\vartheta_2, & \hat{l} &= \vartheta_3 \end{aligned} \quad (4.19)$$

and final operator in coordinate $(\theta, \theta_1, \theta_2, \theta_3, \vartheta, \vartheta_1, \vartheta_2, \vartheta_3)$ is

$$\mathbf{D}_0 = \begin{bmatrix} -\partial & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\partial & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\partial & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (4.20)$$

Then, the composition of (4.16-17) with(4.19) supplys us free field realization for the $W_4^{(2)}$ algebra(3.7).

Remarks:

- (1). This construction provides us, as a by-product, a new proof of the Hamiltonian nature of the structure(3.4);
- (2).The modified hierarchies for each coordinates are easily calculated;
- (3).With the free field realizations, we may construct quantized algebras for the Poisson algebra(2.1), (3.4) and(3.7).

5 TODA TYPE THEORIES CONNECTED WITH $W_4^{(2)}$

In this section we shall construct the Toda type theory connected with the $W_4^{(2)}$ algebra. Exactly speaking, we shall construct a Toda theory which corresponds to two copies of the $W_4^{(2)}$ algebra: one copy is holomorphic, the other is anti-holomorphic. The construction is based on the following observations. Recall that the W -basis of the holomorphic copy of $W_4^{(2)}$ used in [7] is arranged in the following Drinfeld-Sokolov gauge,

$$Q = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ T_1 & G^{(+)} & U & Z \\ Y & T_2 & G^{(-)} & -U \end{pmatrix}.$$

Similarly we can have a W -basis of the anti-holomorphic copy of $W_4^{(2)}$ which can also be arranged into the Drinfeld-Sokolov gauge

$$\bar{Q} = \begin{pmatrix} 0 & 0 & \bar{T}_1 & \bar{Y} \\ 0 & 0 & \bar{G}^{(+)} & \bar{T}_2 \\ -1 & 0 & \bar{U} & \bar{G}^{(-)} \\ 0 & -1 & \bar{Z} & -\bar{U} \end{pmatrix}.$$

Let g be the solution of the following linear systems,

$$\partial_+ g + Qg = 0, \quad \partial_- g + g\bar{Q} = 0.$$

We can easily see that the matrix g can be realized by the matrix elements

$$g_a^b = \sum_i f_a^i \bar{f}_i^b,$$

where f_i^j and \bar{f}_i^j satisfy

$$\partial_x f_a^j = -f_{a+2}^j, \quad \partial_x \bar{f}_j^b = -\bar{f}_j^{b+2}, \quad a = 1, 2.$$

and \bar{f}_i^b have the similar property. Define the main diagonal subdeterminants Δ_a of the matrix g , *i.e.*

$$\Delta_a = \begin{vmatrix} g_1^1 & \dots & g_1^a \\ \vdots & & \vdots \\ g_a^1 & \dots & g_a^a \end{vmatrix},$$

and, in particular, $\Delta_0 \equiv 1$, we can prove, by tedious but direct calculations, that the matrix T with the elements (here $\Delta_a(i, j)$ denotes the algebraic co-minor of Δ_a with respect to g_i^j)

$$T_a^b \equiv \sqrt{\frac{\Delta_{a-1}}{\Delta_a}} \sum_{l=1}^a \frac{\Delta_a(l, a)}{\Delta_{a-1}} f_a^b$$

satisfy the following equations,

$$\partial_{\pm} T = \pm \left(\frac{1}{2} \partial_{\pm} \Phi + \exp(\mp \frac{1}{2} \text{ad} \Phi) (\Psi_{\pm} + \mu_{\pm}) \right) T, \quad (5.1)$$

where we used the following abbreviations of notations,

$$\begin{aligned}
\Phi &= \sum_{i=1}^3 \phi^i H_i, \quad \phi^a = \ln \Delta_a, \\
\Psi_+ &= \sum_{j=1}^3 \sum_{i=1}^4 \text{sign}(i-j) \psi_i^+ A_{ij} E_j, \quad \psi_a^+ = \frac{\Delta_{a+1}(a, a+1)}{\Delta_a}, \\
\Psi_- &= \sum_{j=1}^3 \sum_{i=1}^4 \text{sign}(i-j) \psi_i^- A_{ij} F_j, \quad \psi_a^- = \frac{\Delta_{a+1}(a+1, a)}{\Delta_a},
\end{aligned} \tag{5.2}$$

H_i , E_i and F_i are the standard Chevalley generators of the Lie algebra A_3 written in the defining representation, A is the following matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

and μ_{\pm} are defined as

$$\begin{aligned}
\mu_+ &= \frac{1}{2} \sum_{i,j=1}^3 [E_i, E_j], \\
\mu_- &= -\frac{1}{2} \sum_{i,j=1}^3 [F_i, F_j].
\end{aligned}$$

Equation (5.1) can be viewed as the Lax pair of $W_4^{(2)}$ Toda theory, with the explicit solution of the Toda fields given by equation (5.2). The Toda field equation can be easily obtained from the compatibility condition of the Lax pair (5.1). The result reads

$$\begin{aligned}
\partial_+ \partial_- \Phi + [\text{e}^{\text{ad}\Phi}(\Psi_-), \Psi_+] + [\text{e}^{\text{ad}\Phi}(\mu_-), \mu_+] &= 0, \\
\partial_- \Psi_+ - [\mu_+, \text{e}^{\text{ad}\Phi}(\Psi_-)] &= 0, \\
\partial_+ \Psi_- - [\text{e}^{\text{ad}\Phi}(\Psi_+), \mu_-] &= 0.
\end{aligned}$$

In terms of the component fields, the above equations read (K is the Cartan matrix of A_3)

$$\begin{aligned}
\partial_+ \partial_- \phi^j - \sum_{i,k=1}^4 \text{sign}(i-j) \text{sign}(k-j) \psi_i^+ \psi_k^- A_{ij} A_{kj} \omega^j + \sum_{l=1}^3 \omega^l \omega^j K_{ij} &= 0, \\
\partial_- \psi_j^+ - \sum_{k=1}^4 \text{sign}(k-j) \psi_k^- A_{kj} \omega^j &= 0,
\end{aligned}$$

$$\begin{aligned}
\partial_+ \psi_j^- - \sum_{k=1}^4 \text{sign}(k-j) \psi_k^+ A_{kj} \omega^j &= 0, \\
\omega^j &\equiv \exp\left(-\sum_{i=1}^3 \phi^i K_{ij}\right), \quad (j = 1, 2, 3) \\
\partial_- \psi_4^+ &= \partial_+ \psi_4^- = 0.
\end{aligned}$$

Remarks.

(1). The above construction of Toda type theory is essentially an extension of the technique of W -surfaces, which was first developed by Gervais and Matsuo [20] in the standard W_N cases. Thus the construction given here not only present the $W_4^{(2)}$ Toda equation but also the $W_4^{(2)}$ surface in the sence of [20].

(2). Toda type equations associated with general $W_N^{(2)}$ algebras are already studied by one of the authors (LC) and collaborators in several papers [18]. However those equations restricted to the case of $N = 4$ lack the fields ψ_4^\pm , thus does not really corresponds to $W_4^{(2)}$ algebra. The present equations overcome this shortcoming.

(3). The functions f_a^i and \bar{f}_i^a can be shown to satisfy two commuting families of classical exchange algebra for $a = 1, 2$. For example, the holomorphic family of exchange algebra reads

$$\left\{ f_a^i(x), f_b^j(y) \right\} = -\frac{1}{8} f_a^i(x) f_b^j(y) \text{sign}(x-y) + f_a^j(x) f_b^i(y) [\theta(i-j)\theta(x-y) - \theta(j-i)\theta(y-x)], \quad (5.3)$$

$a, b = 1, 2,$

where

$$\theta(a-b) \equiv \begin{cases} \frac{1}{2} & (a-b=0) \\ 0 & (a-b<0) \\ 1 & (a-b>0) \end{cases}, \quad \text{sign}(a-b) = \theta(a-b) - \theta(b-a). \quad (5.4)$$

Such exchange algebras can be used to reconstruct $W_4^{(2)}$ algebra since one can always write the W -basis of $W_4^{(2)}$ algebra in terms of appropriate determinants consisted of the above functions. This construction of W algebras can also be extended to any classical $W_N^{(l)}$ algebras [19]. Since the classical exchange algebra is the origin of quantum group, it may also be possible to relate quantum W algebras and quantum groups in terms of a quantized version of such constructions.

(4). The canonical Poisson structure for the $W_4^{(2)}$ Toda fields can also be obtained from the exchange relation (5.3) and the explicit solution (5.2) of the field equations.

6 CONCLUSIONS

In this paper we constructed both the integrable evolution equations and the corresponding Toda theory associated to the $W_4^{(2)}$ algebra. Miura maps are presented in connection with the $W_4^{(2)}$ evolution equations, which in turn give a free field realization of $W_4^{(2)}$ algebra.

Though the problems considered here is only a specific case of the W -algebra–evolution equation–Toda system connections, the constructions presented here again assures the widely adopted conjecture that given a W -algebra there must exist an associated system of evolution equations and a corresponding Toda theory.

Besides what have been considered in the main text of this paper, we would like to mention that there are still some unsolved problems such as the connection between the variables appeared in the evolution equations and the Toda fields. As the $W_4^{(2)}$ algebra is much more complicated than the standard W_N series, one should feel reasonable that such connections are not so straightforward as in the standard case.

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